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FINDING A SMALL 3-CONNECTED MINOR MAINTAINING A FIXED MINOR AND A FIXED ELEMENT

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Let N and M be 3-connected matroids, where N is a minor of M on at least 4 elements, and let e be an element of M and not of N. Then, there exists a 3-connected minor $\bar{M}$ of M that uses e, has N as a minor, and has at most 4 elements more than N. This result generalizes a theorem of Truemper and can be used to prove Seymour's 2-roundedness theorem, as well as a result of Oxley on triples in nonbinary matroids.

1. Introduction

Familiarity with matroid theory is assumed. For an introduction see [2, 12].

Let M be a matroid on $E(M)$ (or simply E), with Whitney rank function r. A bipartition $\{A, B\}$ of E is a (Tutte) *k-separation* [11], for some positive integer k, if $|A|\geqq k\leqq|B|$ (where $|A|$ is the cardinality of A), and $r(A)+r(B)\leqq r(E)+k-1$. M is *n-connected,* for some integer $n\geqq 2$, if M has no k-separation for $k<n$. A 2-connected matroid is called *connected.*

The main theorem of this paper can now be stated.

Theorem 1.1. *Let N and M be 3-connected matroids such that N is a minor of M and $|E(N)|\geqq 4$. If $e\in E(M)-E(N)$, then there exists a 3-connected minor $\bar{M}$ of M such that $e\in E(\bar{M})$, N is a minor of $\bar{M}$, and $|E(\bar{M})-E(N)|\leqq 4$.*

Note that isomorphism is not allowed. Indeed, it is not difficult to show that M has a 3-connected minor $\bar{M}$, with $e\in E(\bar{M})$, such that $\bar{M}$ has an isomorphic copy N' of N as a minor and has *at most one* element more than N'. In the case of Theorem 1.1, the bound of 4 is tight.

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Theorem 1.1 provides necessary and sufficient conditions to force any number of elements into a minor from a specified class of 3-connected matroids. (See Theorem 6.3.) In particular, an easy proof of the following theorem of Seymour [8] is obtained, where U_4^2 is the uniform matroid on four elements in which every pair of elements is a base.

Theorem 1.2. *If M is a 3-connected non-binary matroid, then every pair of elements is contained in the element set of some U_4^2 minor of M.* ∎

Further, in [9] Seymour derives (1.2) from his general so-called "2-roundedness" Theorem (Theorem 6.2), which also follows easily from (1.1). A recent result of Oxley [6] about triples of elements in non-binary 3-connected matroids is also obtained (Theorem 6.5).

The following theorem of Truemper [10] also follows from (1.1):

Theorem 1.3. *If N and M are 3-connected matroids such that N is a minor of M and $|E(N)|\geqq 4$, then M has a 3-connected minor $\overline{M}$ such that N is a proper minor of $\overline{M}$ and $|E(\overline{M})-E(N)|\leqq 3$.* ∎

Indeed, this theorem motivated the conjecture of (1.1); moreover, an alternative proof of (1.3) is contained in the proof of (1.1).

The proof is of (1.1) divided into two parts and presented in sections 3 and 4. In both cases, it is assumed that M is minimal with respect to the given minor N and element e. Section 3 treats the case where N is uniquely expressible as a minor of M, and section 4 assumes the existence of some element both the deletion and contraction of which preserves the minor N.

Section 2 introduces the necessary notation, definitions, and preliminary results, many of which are standard. Theorem 5.1 of section 5 summarizes the possible structures a minimal M can have (it is really this theorem, of which Theorem 1.1 is an immediate corollary, that is being proved in sections 3 and 4). This list is used to prove the theorems in section 6, the applications section.

2. Preliminaries

Let M be a matroid on E with (Whitney) rank function r. M^* denotes the *dual* of M, with rank function r^* given by (for $A\subseteq E$) $r^*(A)=|A|-r(E)+r(E-A)$. A *loop* of M is a one-element circuit, and a *coloop* is a one-element cocircuit. Distinct elements $e,f\in E$ are *parallel* if $\{e,f\}$ is a circuit, and *in series* if $\{e,f\}$ is a cocircuit. The *parallel (series) class* of a fixed element is that element together with all element parallel to (in series with) it. A *triangle* is a 3-element circuit and a *triad* a 3-elements cocircuit. For C a circuit and C^* a cocircuit, the property that $|C\cap C^*|\neq 1$ is called *orthogonality*.

For $X\subseteq E$, $M\setminus X$ denotes the *deletion* of X and $M/X=(M^*\setminus X)^*$ the *contraction*. For disjoint $X, Y\subseteq E$, $N=M\setminus X/Y=M/Y\setminus X$ is a *minor* of M. This will be written as $N\subseteq M$ (or $N\subset M$, if $X\cup Y\neq\emptyset$).

To *simplify* a matroid M means to delete all loops and delete all but one element from each parallel class. *Cosimplification* is the dual of simplification. For $e\in E$,

$M-e$ is used to denote "the" cosimplification of $M\setminus e$ and $M|e$ to denote "the" simplification of M/e.

Given integers $0\leqq n\leqq m\neq 0$, U_m^n denotes the *uniform* matroid on m elements in which every n-element subset is a base. For a graph G, $\mathcal{M}(G)$ denotes the usual polygon matroid of G. Let $n\geqq 3$ be an integer, and let H_n be the graph on $n+1$ nodes in which n of these nodes form a polygon, P, and the remaining node is joined to the nodes of P by single edges. H_n has $2n$ edges and is called a *wheel*. The corresponding matroid, $\mathcal{M}(H_n)$ is also called a wheel. The *whirl* matroid is obtained from $\mathcal{M}(H_n)$ by declaring P to be independent (and leaving all remaining independent sets the same).

The definition of a k-separation is given in section 1. A k-separation $\{A, B\}$ is called *minimal* if $\min\{|A|, |B|\}=k$.

Proofs of the first four lemmas are left to the reader. (Throughout, assume M is a matroid on E.)

Lemma 2.1. *For a bipartition $\{A, B\}$ of E, $r(A)+r(B)-r(E)=r^*(A)+r^*(B)-r^*(E)=r(A)+r^*(A)-|A|$.* ∎

Thus, connectivity is invariant under duality.

Lemma 2.2. *Every minimal k-separation of a k-connected matroid is either a circuit and coindependent or a cocircuit and independent.* ∎

Lemma 2.3. *If M has a non-minimal k-separation $\{A, B\}$, and X is a circuit or cocircuit with $X\cap B=\{x\}$, then $\{A\cup\{x\}, B-\{x\}\}$ is a k-separation of M.* ∎

Lemma 2.4. *Assume M is 3-connected, and let $A\subseteq E$ be such that $|E-A|\geqq 2$. Then the following three statements hold:*

(a) *If $|A|=3$, then A does not include a triangle and a triad.*
(b) *If $|A|=4$, then A does not include 2 triangles and a triad or 2 triads and a triangle.*
(c) *If $|A|=5$, then A does not include 2 triangles and 2 triads.* ∎

Two pairs of sets $\{A, B\}$ and $\{C, D\}$ *cross* if each of the sets $A\cap C$, $A\cap D$, $B\cap C$, $B\cap D$ is nonempty.

Lemma 2.5. ([3]) *Assume M is 3-connected and let $e\in E$. Then every 2-separation of $M\setminus e$ crosses every 2-separation of M/e, and one of these two matroids has only minimal 2-separations; moreover, either $M-e$ or $M|e$ is 3-connected.* ∎

The next lemma follows easily from (2.5) and (2.3).

Lemma 2.6. *Assume M is 3-connected and elements x, y, z, w are distinct such that $\{x, y, z\}$ is a triangle (triad) and either $\{x, y, w\}$ or $\{x, y, w, z\}$ is a cocircuit (circuit). Then $M\setminus z$ (M/z) has only minimal 2-separations.* ∎

Lemma 2.7. (Tutte [11]) *Assume M is 3-connected and $\{e, f, g\}$ is a triangle (triad) of M such that $M\setminus e$ and $M\setminus f$ (M/e and M/f) are 2-separable. Then e is in a triad (triangle) with exactly one of f, g.* ∎

The next lemma can be proved by a straightforward application of (2.1).

Lemma 2.8. *Let N be a 3-connected minor of a matroid M, and let $\{A, B\}$ be a k-separation of $M, k \leqq 2$. Then*

$$\min \{|A \cap E(N)|, |B \cap E(N)|\} \leqq k-1. \quad \blacksquare$$

Lemma 2.9. *If $\{e, f, g\}$ is both a triangle and a triad of M, then $M \setminus e/f = M \setminus f/e$.*

Proof. The proof is an easy exercise in circuit elimination. ▮

Assume $N \subset M$, both N and M are 3-connected, and $e \in E(M) - E(N)$. The element e is called *removable* if either $N \subseteqq M \setminus e$ and $M \setminus e$ is 3-connected or $N \subseteqq M/e$ and M/e is 3-connected. A pair of elements, (y, x) is a *removable pair* if neither y nor x is removable, $N \subseteqq M/y \setminus x$, and $M/y \setminus x$ is 3-connected. An element e is called *indifferent* if $N \subseteqq M \setminus e$ and $N \subseteqq M/e$. The following lemma is an easy generalization of Lemma 3.1 of [4].

Lemma 2.10. *Assume $N \subset M$ are both 3-connected matroids and $E(N) \geqq 4$. Then the following statements hold.*

(a) *Where $X = \{x \in E\colon N \subseteqq M \setminus x, N \nsubseteqq M/x\}$ and $Y = \{y \in E\colon N \subseteqq M/y, N \nsubseteqq M \setminus y\}$, no triangle (triad) of M contains two elements of Y (X).*

(b) *If $N \subset M \setminus e$ (M/e) and $M \setminus e$ (M/e) has a 2-separation $\{A, B\}$ such that A contains the set of indifferent elements of $M \setminus e$ (M/e) and $|B \cap E(N)| \leqq 1$, then e is in a triad (triangle) of M.* ▮

Theorem 2.11. ([4]) *Assume $N \subset M$ are both 3-connected matroids, $|E(N)| \geqq 4$, and there are unique subsets X, Y of $E(M)$ such that $N = M \setminus X/Y$. (There are no indifferent elements.) Assume M has no removable element. Then each $x \in X (y \in Y)$ is in a removable pair. Moreover, if (y, x) is a removable pair, then $M/y \setminus x$ has no removable element.* ▮

Lemma 2.12. *If N is a minor of a connected matroid M, then M does not have exactly one indifferent element.*

Proof. Suppose e is the unique indifferent element of M. Let

$$X = \{x \in E(M) - (E(N) \cup \{e\})\colon N \subset M \setminus x\}$$

and

$$Y = \{y \in E(M) - (E(N) \cup \{e\})\colon N \subset M/y\}.$$

Clearly $\{X, Y\}$ partitions $E(M) - (E(N) \cup \{e\})$, and letting $M_1 = M \setminus X/Y$, we have $N = M_1 \setminus e = M_1/e$. It follows that e is either a loop or a coloop of M_1. Without loss of generality, assume e is a loop of M_1. Since e is not a loop of M, there exists a nonempty subset $Y_1 \subseteqq Y$ such that $\{e\} \cup Y_1$ is a parallel class of $M_2 = M \setminus X/(Y - Y_1)$. But then $N = M_2/e \setminus Y_1$, contradicting the uniqueness of e. ▮

Lemmas 2.13 and 2.14 appear in [10]. The proofs are left to the reader.

Lemma 2.13. *If $M \setminus e (M/e)$ is 3-connected for some $e \in E$, but M is not 3-connected, then e is a loop, coloop, or parallel (series) element of M.* ▮

Lemma 2.14. *Let N be a 3-connected minor of M such that $|E(N)| \geqq 4$ and $|E(M) - E(N)| = 2$. Then M is 3-connected and there is no 3-connected matroid $\overline{M}$ such that $N \subset \overline{M} \subset M$ if and only if $N = M \setminus e/f$, for some elements $e \neq f$, and there* *exist distinct elements* $n, m \in E(N)$ *such that* $\{e, f, n\}$ *is the unique triangle of M containing f and* $\{e, f, m\}$ *is the unique triad of M containing e.* ▮

Lemma 2.15. *If N is a 3-connected minor of M such that* $|E(N)| \geqq 4$,

$$N = M \setminus (X \cup \{e\})/(Y \cup \{f\})$$

for distinct elements e, f, and there exist distinct elements $n, m \in E(N)$ *such that* $\{e, f, n\}$ *is a triangle and* $\{e, f, m\}$ *is a triad of M, then* $M \setminus X/Y$ *is 3-connected.*

Proof. By (2.14), it suffices to show that $\{e, f, n\}$ is a triangle and $\{e, f, m\}$ is a triad of $M_1 = M \setminus X/Y$. Now, $\{e, f, n\}$ is the union of circuits of M_1, one of which, C say, contains n. Since n is not a loop, $e \in C$ or $f \in C$. But if $|C \cap \{e, f\}| = 1$, then orthogonality is violated, since each of e, f is contained in a cocircuit of M_1 contained in $\{e, f, m\}$. Thus, $C = \{e, f, n\}$. Similarly, $\{e, f, m\}$ is a cocircuit. ▮

Lemma 2.16. *If N is a 3-connected minor of M such that* $|E(N)| \geqq 4$, $N \subseteqq M/e \setminus \{f, g\}$, *and there exist distinct* $n, m \in E(N)$ *such that* $\{e, f, n\}$ *and* $\{e, g, m\}$ *are triangles of M, then there exist* $X, Y \subseteqq E(M)$ *such that* $N = M/(\{e\} \cup Y) \setminus (\{f, g\} \cup X)$ *and* $M_1 = M \setminus X/Y$ *is 3-connected.*

Proof. Let $X, Y \subseteqq E(M)$ be such that $\{e\} \cup Y$ is independent, and $N = M/(\{e\} \cup Y) \setminus (\{f, g\} \cup X)$. We first show that $\{e, f, n\}$, $\{e, g, m\}$, are triangles of M_1. Since $\{e\} \cup Y$ is independent, e is not a loop of M_1. But if $\{e, f, n\}$ is not a triangle of M_1, then e is parallel to n or n is a loop, contradicting in either case, 3-connectivity of N. Clearly M_1 is connected. Suppose M_1 has parallel elements. Then f or g must be parallel to something. Suppose f, p are parallel, for some $p \in E(N)$. Then p, n are parallel in $N \subset M_1/e$, a contradiction. If f, g are parallel, then n, m are parallel in M/e, again a contradiction. So M_1 has no parallel elements. M_1 has no series elements because the triangles $\{e, f, n\}$, $\{e, g, m\}$ preclude the possibility for e, f or g. Thus by (2.3) and (2.8), if M_1 is not 3-connected, then M_1 has a 2-separation $\{A, B\}$ with, say, $\{e, f, n\} \subseteqq A$ and $|E(N) \cap A| = 1$. If $g \in A$, then $\{A \cup \{m\}, B - \{m\}\}$ is a 2-separation, contradicting (2.8). Thus, $A = \{e, f, n\}$, implying $\{e, f, n\}$ is codependent. By orthogonality with $\{e, g, m\}$ and $\{e, f, n\}$, $\{f, n\}$ is a cocircuit, contradicting 3-connectivity of N. Thus, M_1 is 3-connected. ▮

3. Proof of Theorem 1.1—the uniqueness case

In this section, assume X and Y are unique such that $N = M \setminus X/Y$; that is, for each $x \in X$ ($y \in Y$) N is not a minor of M/x ($M \setminus y$). The proof of Theorem 1.1 in this case is given at the end of this section. It follows easily from Lemmas 3.1—3.5. In (3.1)—(3.5), assume that M and N are 3-connected, $N = M \setminus X/Y$ for X and Y unique, M has no removable pair, and there is an element $e \in X$ that is the unique removable element of M. Let $Z = E(M) - E(N)$.

Lemma 3.1. *If* $M \setminus e$ *has a removable pair,* (y, x), *then the following two statements hold.*

(a) $Z = \{e, x, y\}$.

(b) *There exist distinct elements* $w, z \in E(N)$ *such that* $\{e, w, y\}$ *and* $\{x, y, z\}$ *are the only triangles of M containing y and* $\{x, y, w\}$ *is the only triad of M containing x and the only triad of* $M \setminus e$ *containing x.*

Proof. By (2.14), there exist distinct elements $w, z \in E(M \setminus \{x, e\}/y)$ such that $\{x, y, w\}$ is the only triad of $M \setminus e$ containing x and $\{x, y, z\}$ is the only triangle of $M \setminus e$ containing y. By (2.10), x is in some triad T of M. Since $M \setminus e$ is 3-connected, T is a triad of $M \setminus e$ and therefore $T = \{x, y, w\}$. Hence, $w \notin X$.

Since M has no removable pair, $M \setminus x/y$ is not 3-connected. But $M \setminus x/y \setminus e$ is 3-connected; therefore, by (2.13) and 3-connectivity of M, e is parallel in $M \setminus x/y$, implying e and y are in a triangle of M. By orthogonality with T, this triangle is $\{e, w, y\}$ or $\{e, x, y\}$. If $\{e, x, y\}$ is a triangle, elimination with $\{x, y, z\}$ provides triangle $\{e, y, z\}$, contradicting orthogonality with T. Thus, $\{e, w, y\}$ is the only triangle of M containing e and y. Now, $w \in E(N)$ by (2.10) (a), since $w \notin X$. Suppose $z \notin E(N)$. Then $z \in X$, by (2.10) and triangle $\{x, y, z\}$, and therefore by (2.10) (b), z is in some triad of M. By (2.10) and orthogonality, this triad cannot contain x and must contain y and one of e, w, and therefore must be completely contained in $\{e, w, y, x, z\}$, contradicting 3-connectivity of M, by (2.4). Thus $z \in E(N)$, and (b) is proved.

Now, (2.16) implies that $M_1 = M \setminus (X - \{e, x\})/(Y - \{y\})$ is 3-connected. If $Z \neq \{e, x, y\}$, then (2.11) implies that M has a removable pair or removable element with respect to M_1, contradicting the assumption that e is the unique removable element and M has no removable pair with respect to N. ∎

Lemma 3.2. *If $M \setminus e$ has no removable pair, then the following two statements hold.*

(a) *$M \setminus e$ has a unique removable element, f, and $f \in Y$.*

(b) *There exists an element $n \in E(N)$ such that $\{e, f, n\}$ is the only triangle of M containing f.*

Proof. By (2.11), $M \setminus e$ has a removable element, f. Suppose $f \in X$. Then by (2.13) and the uniqueness of e, e is a loop, coloop or a parallel element in $M \setminus f$, contradicting 3-connectivity of M. Therefore, $f \in Y$, as desired, and by (2.13), e and f are in a triangle $\{e, f, n\}$ of M. It must be that $n \in E(N)$, for otherwise, $n \in X$, by (2.10), and $M/f \setminus n \cong M/f \setminus e$, implying either that (f, n) is a removable pair or n is a removable element. $\{e, f, n\}$ is the only triangle of M containing f, since $M \setminus e/f$ is 3-connected. Now suppose $M \setminus e$ has another removable element $h \neq f$. Then, similarly, $h \in Y$ and $\{e, h, k\}$ is a triangle for some $k \in E(N)$. Circuit elimination implies n is in a circuit of M contained in $\{f, h, k, n\}$, implying n is in a circuit of N contained in $\{k, n\}$, contradicting 3-connectivity of N. ∎

In (3.3)—(3.5), assume $M \setminus e$ has no removable pair, and that f, n are as in (3.2).

Lemma 3.3. *If $M \setminus e/f$ has a removable pair, (y, x), then the following statements hold.*

(a) $Z = \{e, f, x, y\}$.

(b) *There exists an element $z \in E(N)$ distinct from n such that:*

(i) *$\{x, y, z\}$ is a triad of M, and is the only triad of $M \setminus e/f$ containing x,*

(ii) *$\{x, y, n\}$ is a triangle of M, the only triangle of M containing y, and the only triangle of $M \setminus e/f$ containing y,*

(iii) *$\{f, x, n\}$ is a triad of M.*

Proof. Applying (3.1) to $N \subset M \setminus e$ and using duality, it is immediate that $Z = \{e, f, x, y\}$ and there are distinct elements $w, z \in E(N)$ such that $\{f, w, x\}$,

$\{x, y, z\}$ are the only triads of $M\setminus e$ containing x, $\{x, y, z\}$ is the only triad of $M\setminus e/f$ containing x, and $\{x, y, w\}$ is the only triangle of $M\setminus e$ (and of $M\setminus e/f$) containing y.

We first show that $\{x, y, w\}$ is the only triangle of M containing y. Otherwise, $\{y, e, p\}$ is a triangle of M for some p. Elimination with $\{e, f, n\}$ implies that p is in a cricuit contained in $\{f, n, y, p\}$ and thus, p is in a circuit of $M/\{f, y\}$ contained in $\{n, p\}$. By 3-connectivity of $N, p=x$. Thus, $\{y, e, x\}$ is a triangle, implying $\{x, e, w\}$ and $\{y, e, w\}$ are triangles. Now, by (2.10), x is in a triad of M. Since e is in no triad, orthogonality implies that $\{x, w, y\}$ is a triad (and a triangle) of M, a contradiction to 3-connectivity of M.

We next show that $w=n$. First, if $\{f, w, x\}$ is a triad of M, then by orthogonality with $\{e, f, n\}$, $n=w$. If $\{f, w, x\}$ is not a triad, then $\{f, w, x, e\}$ is a cocircuit of M. By (2.10), some triad T of M contains x. If $f\in T$, then $T=\{f, n, x\}$, implying $\{n, x, w, e\}$ contains a cocircuit of M containing n, contradicting 3-connectivity of N. Thus, $f\notin T$ and therefore T is a triad of $M\setminus e/f$, implying $T=\{x, y, z\}$, and also that T is the only triad of M containing x.

Since the unique triangle containing y contains x and the unique triad containing x contains y, $M\setminus x/y$ has no minimal 2-separations. Thus, (2.3) applied first to $\{e, f, n\}$ and then to $\{f, w, e\}$ implies a 2-separation $\{A, B\}$ of $M\setminus x/y$ with $\{e, f, n, w\}\subseteq A$, implying, by (2.8) that $n=w$.

With $n=w$, we have triangle $\{x, y, n\}$ of M and triad $\{f, x, n\}$ of $M\setminus e$. Now if $\{f, x, n, e\}$ is a cocircuit of M, then $\{f, n, e\}$ is a triangle and a triad of $M\setminus x$. But then by (2.9), $M\setminus\{x, e\}/f=M\setminus\{x, f\}/e$, a contradiction to the uniqueness of X and Y.

It remains to show that $\{x, y, z\}$ is a triad of M. Otherwise, $\{x, y, z, e\}$ is a cocircuit, implying by orthogonality that $z=n=w$, a contradiction. (z and w were distinct). ∎

In Lemmas 3.4, 3.5, assume $M\setminus e/f$ has no removable pair of elements.

Lemma 3.4. *$M\setminus e/f$ has a unique removable element, $g\in X$, and $\{f, g, n\}$ is a triad of M, the only triad of M containing g. Moreover, $M\setminus e/f\setminus g$ has no removable pair of elements.*

Proof. Applying (3.2) and duality to $M\setminus e$, $M\setminus e/f$ has a unique removable element g, $g\in X$, and there exists an element $m\in E(N)$ such that $\{f, g, m\}$ is a triad of $M\setminus e$, the only triad of $M\setminus e$ containing g. By (2.10), g is in a triad T of M and since e is in no triad, T is a triad of $M\setminus e$ and therefore, $T=\{f, g, m\}$. By orthogonality with $\{e, f, n\}$, $n=m$.

Suppose $M\setminus e/f\setminus g$ has a removable pair, (y, x). Then applying (3.3) and duality to $M\setminus e$, $\{x, y, n\}$ is a triad of $M\setminus e$, the only triad of $M\setminus e$ containing x. Since x is in a triad of M, $\{x, y, n\}$ is a triad of M, contradicting orthogonality with $\{e, f, n\}$. ∎

Let $g\in X$ be the unique removable element of $M\setminus e/f$.

Lemma 3.5. *If $N\subset M\setminus e/f\setminus g$, then $Z=\{e, f, g, h\}$, where $h\in Y$ and $\{g, h, n\}$ is a triangle of M.*

Proof. By (3.4), $M\setminus e/f\setminus g$ has no removable pair. Hence, applying (3.4) and duality, $M\setminus e/f\setminus g$ has a unique removable element $h\in Y$, and $\{g, h, n\}$ is a triangle of $M\setminus e$, the only triangle of $M\setminus e$ containing h. Suppose $\{g, h, n\}$ is not the only triangle of M containing h. Then $\{e, h, p\}$ is a triangle of M for some p. Elimination with $\{e, f, n\}$ implies that $\{h, p, f, n\}$ includes a circuit C containing n. It follows that $p=g$ and $C=\{f, g, n\}$ is a triangle and triad, contradicting 3-connectivity. Therefore, $\{g, h, n\}$ is the only triangle of M containing g.

Suppose $|Z|>4$. Then applying (3.4) again, $M\setminus e/f\setminus g/h$ has no removable pair and has a removable element $k\in X$, and $\{h, k, n\}$ is a triad of $M\setminus e/f$. Further, as in the previous paragraph, (dually) $\{h, k, n\}$ is the only triad of $M\setminus e$ containing k. Since k is contained in some triad of M, it follows that $\{h, k, n\}$ is a triad of M, contradicting orthogonality with $\{e, f, n\}$. ▌

To complete the proof of Theorem 1.1 in the uniqueness case, assume N and M are 3-connected matroids, $N=M\setminus X/Y$ (X and Y unique) $|E(N)|\geqq 4$, and $e\in E(M)-E(N)$. In addition, assume M is minimal with respect to N and e; that is, M has no 3-connected proper minor $\bar{M}$ with $N\subset\bar{M}$ and $e\in E(\bar{M})$. Let $Z=X\cup Y$. By duality, assume $e\in X$. If $|Z|=1$, then there is nothing to prove. If $|Z|=2$, then $Z=\{e, f\}$, say, and by (2.13) and minimality of M, $f\in Y$ and e and f are in a triangle with some element $n\in E(N)$. Assume $|Z|>2$. If e is not a removable element, then by (2.11) there is a removable pair (y, x), $x\neq e$, contradicting minimality of M. Thus e is a removable element, the only removable element of M, and by minimality of M, M has no removable pair. The theorem now follows from (3.1)—(3.5).

4. Proof of Theorem 1.1—the non-uniqueness case

Assume N and M are 3-connected matroids, N is a minor of M, $|E(N)|\geqq 4$, and $e\in E(M)-E(N)$. Assume M is minimal with respect to N and e.

In this section, assume M has an indifferent element. By (2.12), there is an indifferent element $f\neq e$. Using duality, if necessary, and (2.5), assume that M/f has only minimal 2-separations. Since $N\subseteqq M/f$ and M/f is 3-connected, minimality of M implies that e and f are in a triangle, $\{e, f, n\}$, for some $n\in E(N)$. Therefore, $N\subset M\setminus e$. Lemmas 4.1—4.5 complete the proof of the theorem. Let

$$Z = E(M)-E(N).$$

Lemma 4.1. *For every triad T of M, $e\in T$ if and only if $f\in T$.*

Proof. If $f\in T$ and $e\notin T$, then by orthogonality with $\{e, f, n\}$, $T=\{f, g, n\}$, for some element g, implying $N\subset M/g$. But by (2.6), M/g has only minimal 2-separations. By minimality of M, $\{g, e, m\}$ is a triangle for some $m\in E(N)$. By orthogonality, $n=m$, contradicting 3-connectivity, by (2.4).

If $e\in T$ and $f\notin T$, then $T=\{e, n, g\}$, again implying $N\subset M/g$ and $\{g, e, m\}$ is a triangle for some $m\in E(N)$. By 3-connectivity of M, $m\neq n$. Let $M_1=M/f\setminus e$. Since g and n are in series in M_1, $N\subseteqq M_1/g$. But m, n are parallel in M_1/g, contradicting 3-connectivity of N. ▌

Lemma 4.2. (a) *If $g\neq e$ is an indifferent element of M, then M/g has only minimal 2-separations, and $\{e, g, m\}$ is a triangle of M, for some $m\in E(N)$.*

(b) *If* $g \neq h$ *are indifferent elements and* $\{e, g, m\}$, $\{e, h, p\}$ *are triangles of* M *for* $m, p \in E(N)$, *then* $m \neq p$.

(c) *If* $g \neq e, f$ *is an indifferent element, then* $M \setminus g$ *has a nonminimal 2-separation.*

Proof. Let $g \neq e$ be an indifferent element. If $g=f$, (a) is true by assumption, so assume $g \neq f$. By (2.5), $M \setminus g$ or M/g has only minimal 2-separations. If it is $M \setminus g$, then minimality of M implies $\{g, e, m\}$ is a triad for some $m \in E(N)$, contradicting (4.1). This proves (c), and (a) follows by minimality of M. To prove (b), suppose $m=p$. Then, e, m, h are parallel in M/g, contradicting (a). ∎

Lemma 4.3. *If element* f *is in a triad, then* $|Z|=2$ *or* 3.

Proof. By (4.1), if f is in a triad, T, then $e \in T$. So $T=\{e, f, g\}$ for some element g. If $g \in E(N)$, then $g \neq n$ and by (2.15) and minimality of M, $Z=\{e, f\}$. If $g \notin E(N)$, then $N \subset M/g$. By (2.6), M/g has only minimal 2-separations. By minimality of M, $\{g, e, m\}$ is a triangle for some $m \in E(N)$. Thus, $N \subseteq M \setminus f/e \setminus g = M \setminus \{e, f, g\}$. Since g is indifferent, $m \neq n$, by (4.2b), and then (2.16) and minimality imply $Z=\{e, f, g\}$. ∎

Note that in (4.3), if $|Z|=3$, then the conditions of (5.1d) are satisfied.

Lemma 4.4. *If either* e, f *are the only indifferent elements of* M *or* M *has at least* 2 *indifferent elements distinct from* e, f, *then* f *is in a triad.*

Proof. Assume f is in no triad. Then $M \setminus f$ is not 3-connected and has only nonminimal 2-separations. Now by (2.12), if e, f are the only indifferent elements of M, $M \setminus f$ has no indifferent elements, and the result follows by (2.10b). Otherwise, (4.2) and (2.3) imply the existence of a 2-separation $\{A, B\}$ of $M \setminus f$ such that $\{e\} \cup W \subset A$, where W is the set of indifferent elements of M, and $|A \cap E(N)| \geqq 2$. By (2.8), $|B \cap E(N)| \leqq 1$, and again (2.10b) implies the result. ∎

Lemma 4.5. *If* g *is the unique indifferent element distinct from* e, f, *then* $Z=\{e, f, g\}$, *and* $\{e, f, g\}$ *is a triad.*

Proof. By minimality of M, $M \setminus f$ has a 2-separation. If $M \setminus f$ has no indifferent element, then f is in a triad, by (2.10b), and then (4.3) implies $Z=\{e, f, g\}$. If $M \setminus f$ has an indifferent element, then by (2.12), e is indifferent in $M \setminus f$ and therefore in M. Let $\{e, g, m\}$, $m \in E(N)$, be a triangle of M, as guaranteed by (4.2). Then $N \subseteq M/e \setminus \{g, f\}$. By (2.16) and minimality, $Z=\{e, f, g\}$, as desired. Moreover, $M \setminus f$ has no indifferent elements, so by (2.10b), f is in a triad. Thus, in either case, f is in a triad T. By orthogonality with triangles $\{e, f, n\}$ and $\{e, g, m\}$, if $T \neq \{e, f, g\}$, then $T=\{e, f, m\}$. But then (2.6) implies that $M \setminus g$ has only minimal 2-separations, implying by (2.10b) that g is in a triad. By orthogonality, this triad is contained in $\{e, f, g, n, m\}$, contradicting 3-connectivity of M, by (2.4). ∎

5. The minimal structures

The purpose of this section is to present a statement of the main theorem that can be applied to prove the results in section 6. The possible forms of M 3-connected and minimal with respect to N and element e are enumerated in the following theorem.

Theorem 5.1. *If $N \subset M$ are 3-connected matroids, $|E(N)| \geqq 4$, $e \in Z = E(M) - E(N)$, and M is minimal with respect to N and e, then one of the following conditions holds (up to duality).*

(a) $|Z| = 1$: $N = M \setminus e$,

(b) $|Z| = 2$: $N = M \setminus e/f$. *For some* $n \in E(N)$, $\{e, f, n\}$ *is a triangle.*

(c) $|Z| = 3$: $N = M \setminus \{e, g\}/f$. *For some* $n \in E(N)$, $\{e, f, n\}$ *is a triangle and* $\{f, g, n\}$ *is a triad.* $M \setminus e$ *is 3-connected.*

(d) $|Z| = 3$: $N = M \setminus \{e, g\}/f = M \setminus \{e, f\}/g = M \setminus \{f, g\}/e = M \setminus \{e, f, g\}$. *For distinct* $n, m \in E(N)$, $\{e, f, g\}$ *is a triad, and* $\{e, g, m\}$ *and* $\{e, f, n\}$ *are triangles.*

(e) $|Z| = 4$: $N = M \setminus \{e, g\}/\{f, h\}$. *For some* $n \in E(N)$, $\{e, f, n\}$ *and* $\{g, h, n\}$ *are triangles, and* $\{f, g, n\}$ *is a triad.* $M \setminus e$ *and* $M \setminus e/f$ *are 3-connected.* ▮

We remark that (d) arises in section 4 (see the comment after Lemma 4.3).

Lemma 5.2. *If M is of the form* (5.1e), *then M has a minor $N' \cong N$ such that $N' \subset M \setminus h/g$ and $M \setminus h/g$ is 3-connected.*

Proof. $M \setminus e/f \cong M \setminus n/f \cong M \setminus n/g \cong M \setminus h/g$. ▮

6. Applications

In this section we introduce a slight generalization of Seymour's definition [8] of a "k-rounded" class of matroids. Seymour's 2-roundedness theorem is proved, using Theorem 1.1, as well as a new roundedness theorem.

For completeness, we begin by stating the analog of Theorem 1.1 in the (2−)connected case. The proof of this theorem is quite easy. (See, for example, [5] or [7].)

Theorem 6.1. *If $N \subset M$ are connected matroids and $e \in E(M) - E(N)$, then there exists a connected matroid $\bar{M} \subseteqq M$ such that $N = \bar{M} \setminus e$ or $N = \bar{M}/e$.*

A class of matroids $\mathscr{F}$ is *(k, l)-rounded*, for positive integers k and l, if the following three conditions hold:

(i) Every $M \in \mathscr{F}$ is k-connected and has at least four elements;

(ii) If $M \in \mathscr{F}$ and $M' \cong M$, then $M' \in \mathscr{F}$;

(iii) If M is a k-connected matroid and $N \subseteqq M$ for some $N \in \mathscr{F}$, then for every $A \subseteqq E(M)$, with $|A| \leqq l$, there exists a matroid $N' \in \mathscr{F}$ such that $N' \subseteqq M$ and $A \subseteqq E(N')$.

Seymour [8] has proved the following theorem, making the task of checking whether a given class of matroids is (2, 1) – or (3, 2) – rounded finite.

Theorem 6.2. *For* $(k, l) = (2, 1)$ *or* $(3, 2)$, *conditions* (i), (ii), (iii) *are equivalent to* (i), (ii), (iv) *where* (iv) *is* (iii) *with the added condition that* $|E(M) - E(N)| \leqq 1$.

Proof. The case $(k, l)=(2, 1)$ follows immediately from (6.1). Hence, we provide the proof for $(k, l)=(3, 2)$. Assume (i), (ii), (iv) hold. We wish to show (iii). Let $N \subset M$ be 3-connected matroids, with $N \in \mathscr{F}$, and let $e, p \in E(M)$. By (6.1), (2.13), (ii) and (iii) we can assume $p \in E(N)$. Assume $e \notin E(N)$, and assume M is minimal with respect to N and e. Now apply (5.1), letting $Z=E(M)-E(N)$. If $|Z|=1$, we are finished. By (5.2), we need only consider the cases where $|Z|=2$ and 3. Moreover, since in these cases there is a matroid $\bar{M}$ in which e and n are parallel and $\bar{M} \setminus e=N$, assume $n=p$.

Case 1. Consider the situation in (5.1b). If $M \setminus e$ is 3-connected, then, by (iv), $M \setminus e$ has a minor $N_1 \in \mathscr{F}$ with $\{f, n\} \subseteq E(N_1)$. Let $N_1=M \setminus X/Y \setminus e$. By (2.13), if $M_1=M \setminus X/Y$ is not 3-connected, e is parallel in M_1 to some element m. Since $\{e, f, n\}$ is a union of circuits in M_1 and $M_1 \setminus e$ is 3-connected, $m \neq n$. Thus $M_1 \setminus m \cong N_1$ and $e, n \in E(M_1 \setminus m)$. If M_1 is 3-connected, (iv) implies the result.

Assume $M \setminus e$ is not 3-connected. By (2.13), $\{e, f, m\}$ is a triad, for some $m \in E(N)-\{n\}$. Then $N=M/f \setminus e \cong M/m \setminus e$. If M/m is 3-connected, apply (iv.) If not, then e and q are parallel, for some $q \neq n$, and $N'=M/m \setminus q \cong N$, as required.

Case 2. In (5.1d) $N \cong M \setminus \{m, f\}/g$, and we are done.

Case 3. In (5.1c), $M \setminus \{e, g\}/f \cong M \setminus \{n, f\}/g$. $\bar{M}=M/g \setminus f$ is 3-connected, we are done; otherwise, exchanging n and e reduces this case to Case 2. ▮

Applying (5.1) and (5.2) inductively, we obtain:

Theorem 6.3. *For* $k=3$, $l \geqq 4$, *conditions* (i), (ii), (iii) *are equivalent to* (i), (ii), (v), *where* (v) *is* (iii) *with the added condition that* $|E(M)-E(N)| \leqq 3$. ▮

For $l=3$, (6.3) can be strengthened.

Theorem 6.4. *For* $(k, l)=(3, 3)$, *conditions* (i), (ii), (iii) *are equivalent to* (i), (ii), (vi), *where* (vi) *is* (iii) *with the added condition that* $|E(M)-E(N)| \leqq 2$.

Proof. Let $N \subset M$ be 3-connected matroids with $N \in \mathscr{F}$, and let $\{e, a, b\} \subseteq E(M)$. By (6.2), we can assume $\{a, b\} \subseteq E(N)$, and by (5.1), (5.2), assume $|E(M)-E(N)|=3$. Moreover, since (5.1c) and (5.1d) are symmetric in e, n, we need only consider case (5.1d) with $m=b$ and $n=a$. By orthogonality and 3-connectivity of M, b is in no triad of M. Thus, by (2.6), $M \setminus b$ is 3-connected. Applying (vi), $N_1=M \setminus X/Y \setminus b$ is an element of $\mathscr{F}$ for some $X, Y \subseteq E(M)$, with $\{n, g, e\} \subseteq E(N_1)$. Clearly, $f \in N_1$. Now it is easy to see that either $M_1=M \setminus X/Y$ is 3-connected, and we can apply (vi), or b, p are parallel in M_1, $p \neq n, e$, and $M_1 \setminus p \in \mathscr{F}$. ▮

Similar to the proof of (6.4), a result of Oxley [6] can be proved using (5.1) and (6.2).

Theorem 6.5. *If M is a non-binary 3-connected matroid, then every triple is either contained in the element set of a minor isomorphic to U_4^2 or is the set of spoke or rim elements of a minor isomorphic to the six-element whirl.* ▮

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